

# On Zeeman Topology in Kaluza-Klein and Gauge Theories

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## Abstract

E. C. Zeeman [1] has criticized the fact that in all articles and books until that moment (1967) the topology employed to work with the Minkowski space was the Euclidean one. He has proposed a new topology, which was generalized for more general space-times by Goebel [2]. In the Zeeman and Goebel topologies for the space-time, the unique continuous curves are polygonals composed by time-like straight lines and geodesics respectively. In his paper, Goebel proposes a topology for which the continuous curves are polygonals composed by motions of charged particles. Here we obtain in a very simple way a generalization of this topology, valid for any gauge fields, by employing the projection theorem of Kaluza-Klein theories (page 144 of Bleecker [3]). This approach relates Zeeman topologies and Kaluza-Klein, therefore Gauge Theories, what brings insights and points in the direction of a completely geometric theory.

## 1 Introduction

In 1967, E. C. Zeeman [1] criticized the topology usually employed to work with the Minkowski space and proposed a substitute. These criticisms were done by Cel'nik [4] in the following year. In all books and articles until then the topology employed for Minkowski space was the Euclidean topology of the 4-dimensional space. This was done in a straight way or implicitly when operations involving limits were performed.

Despite the developments of Zeeman's proposal, done for Lorentzian manifolds by Goebel ([4],[5]), Hawking-King-MacCarthy [6], Domiaty ([7],[8],[9]) and Malament [10], the situation has not changed nowadays, almost all books and articles on Minkowski space and other space-times have employed the Euclidean topology of the 4-dimensional Euclidean Space and its correspondent generalizations. It seems that for the good idea of Zeeman it was not found yet a sounding application.

The topology proposed by Zeeman in his seminal article is the finest topology that induces the 3-dimensional Euclidean topology on every space axis and the 1-dimensional Euclidean topology in every time axis. For this topology the continuous curves in the Minkowski space  $M$  are polygonals composed by straight time-like paths and the group of homeomorphisms of  $M$  becomes the non homogeneous Lorentz group increased by dilatations. The space-like axis is a hypersurface with a time-like normal vector and a time-like axis is a straight time-like line. All the proofs in Zeeman paper remain valid if instead of Minkowski space time he had employed a  $1+n$  dimensional linear semi-Riemannian manifold with one time-like and  $n > 3$  space-like

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dimensions. Then the group of homeomorphisms for the Zeeman topology would be the generalization of the non homogeneous Lorentz group, that is the semidirect product of the  $n$ -dimensional translations with the connected subgroup of  $O(1, n)$  increased by the dilatations. Some generalizations for a space with  $p$  space-like and  $q$  time-like dimensions were studied for  $p$  and  $q$  not equal in [11].

Goebel [3] has generalized the Zeeman topology for more general Lorentzian manifolds, where the continuous curves would be geodesic polygonals, that is, composed by timelike geodetic paths. It is well known that in the Riemannian case [4] we can recover the topology from the metric and geodesics by defining a basis of open normal balls. What Goebel has done was to construct from the Lorentzian metric the local neighborhoods of the Zeeman topology, obtaining its generalization for a Lorentzian manifold. Also the proofs of Goebel paper are all valid if we had as space-time a semi-Riemannian manifold with one time-like and  $n > 3$  space-like dimensions. Goebel also proposed to change the geodesics in his construction by the motion of electrically charged particles, obtaining a topology for which a continuous curve is <<a chain of finitely many connected world lines of freely falling charged test particles>>(page 296 of [3]).

Hawking, King and MacCarthy have proposed another topology [6] for which the continuous paths included all the possible time-like paths, there they argue that this would be physically more interesting, since <<even in general relativity, particles need not move along geodesics since, for example, they may be charged and an electromagnetic field may be present (and this applies to special relativity also)>>. But if we intend to have a completely geometric theory in which all the fields acting on the particle are included in the space-time structure, a particle has no more to do in an empty space-time than follow geodesics, or straight time-like paths if we are in the Minkowski case.

On the other hand, Zeeman topology gives a polygonal composed of many straight time-like paths and not only one. The interpretation given originally by Zeeman [1] as the... <<path of a freely moving particle under a finite number of collisions>>... which was repeated many times in the literature can also be criticized in the same way since, travelling in the empty space, a particle has no target to hit. Analogous problem has some interpretation of Goebel's generalization, where we have polygonals of geodesic paths.

But the situation is not so bad in the Goebel and Zeeman cases since it is possible to differentiate along time-like straight lines and it is possible to define a massive particle's path as a continuous and differentiable path, avoiding the zig-zags. It is enough to discover the wheel, the axiomatic gain is yet very big.

Here, following the idea of obtaining a completely geometric theory, that we turn to Goebel's charged particles topology. We employ the projection theorem for Kaluza-Klein theories (part 2) to present it in a very simple way that works as well to general gauge fields (part 3). This approach relates Zeeman topologies and Gauge Theories, bringing insights and suggesting interesting conjectures and developments (part 4) and from the universality of the gauge theories, it points in the direction of a completely geometric theory.

For this, in what follows we will not deal with polygonals of, but with geodesics, any way the results, valid path by path, could be rewritten in terms of polygonals.

## 2 The Projection Theorem in Gauge and Kaluza-Klein Theories

The Kaluza-Klein theories are the classical model for Gauge Theories. The space-time of such theories is the total space  $P$  of a fiber bundle  $\pi : P \rightarrow M$  (here we shall follow Bleecker, [2]). As seen above it is natural to define on  $P$  a metric which turns it into a semi-Riemannian manifold with signature  $(1, n)$ ,  $n = 3 + \dim G$ , where the number of space-like dimensions is three plus the dimension of the gauge group  $G$  of the theory,

which is also the structure group, diffeomorphic to the fiber of the bundle (here we suppose the Lie group  $G$  compact). The Minkowski space, or any other relativistic space-time  $M$  is the base space of the fiber bundle.

Additionally we suppose we have defined a connection 1-form  $\omega \in \Omega^1(P, \text{Lie } G)$  on  $P$  with values in the Lie algebra of  $G$ ,  $\text{Lie } G$ . This connection gives rise to a covariant derivative  $D^\omega : \Omega^q(P, \dots) \rightarrow \Omega^{q+1}(P, \dots)$  and associated curvature  $\Omega^\omega = D^\omega \omega \in \Omega^2(P, \text{Lie } G)$ . The fiber bundle is locally trivial, that is, there is a cover  $U_\alpha$  for  $M$  and associated local sections  $\sigma_\alpha : U_\alpha \rightarrow P$ , with  $\pi \circ \sigma_\alpha = id_\alpha$ , the identity on  $U_\alpha$ . Each local section is interpreted as a gauge choice, and gives rise to a gauge potential  $iA_\alpha = \sigma_\alpha^* \omega_\alpha \in \Omega^1(U_\alpha, \text{Lie } G)$  and to a gauge field  $iF_\alpha = \sigma_\alpha^* \Omega^\omega \in \Omega^2(U_\alpha, \text{Lie } G)$ .

Under a change of local trivialization or gauge change, determined by the transition function  $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow G$ , we obtain

$$A_\beta = g_{\alpha\beta}^{-1} A_\alpha g_{\alpha\beta} + g_{\alpha\beta}^{-1} dg_{\alpha\beta}, \quad F_\beta = g_{\alpha\beta}^{-1} F_\alpha g_{\alpha\beta}$$

...and if a cover by local trivialization and a set of potentials and fields satisfying the above relations are given, we can reconstruct the connection and the curvature in the principal fiber bundle, furthermore, if the cover and the set of transition function are given, we can rebuild the total space of the fiber bundle.

The electromagnetic theory is a special case of gauge theory where  $G = U(1)$ , this inner degree of freedom being interpreted by the helicity of the photon since Weinberg [13], in such a way that a choice of a set of gauge potentials in  $M$  could be interpreted by the changing from an empty Minkowsky space by one with plenty of photons.

In the general case, the homogeneous field equations, analogous to the Maxwell's homogeneous equations are given by the pull-back, by local sections, of the Bianchi's identity,  $D^\omega \Omega^\omega = 0$ , consequence of Jacobi's identity and Poincaré lemma. These are simply the consequence from the assumption that the field can be derived from a potential.

We now define in the total space of the fiber bundle the natural metric

$$h(X, Y) = \pi^* g(X, Y) + k(\omega(X), \omega(Y)) \quad (1)$$

where  $g$  is the metric in the base space  $M$  and  $k$  is the bi-invariant Killing form on  $G$ . Since this last one is negative definite, the total space has signature  $(1, 3 + \dim G)$  as pointed above.

From the first part of the projection theorem as in page 144 of Bleeker [2] we have, for each geodesic curve  $s \rightarrow \gamma(s)$  in the total space of the bundle with the metric defined in (1), that  $\omega(\gamma'(s)) = Q \in \text{Lie } G$  is a constant momentum in the fiber direction. This is interpreted as the Lie valued gauge charge of the particle.

From the second part of the projection theorem, this geodesic projects onto a gauge charged particle's motion  $x(s) = \pi \circ \gamma(s)$ , for which the velocity  $v = dx/ds$  obeys

$$\frac{Dv}{ds} = [F(x(s))] v(s) \quad (2)$$

where the 4x4 matrix field  $F$  is given in each gauge choice by  $F = k(Q, \Omega_\alpha^\omega)$ . This corresponds to the Lorentz force in the electromagnetic field, and applied to the velocity, gives the Levi-Civita covariant derivative of such velocity. From the transformation rules and from the invariance properties of the Killing form,  $F$  does not depend on the gauge choice and we can write  $F = k(Q, \Omega^\omega)$ .

From the proof of the projection theorem, for each motion of a charged particle in the base manifold and a given  $Lie G$  valued charge  $Q$  we get a unique geodesic in the total space.

In what follows we consider the Goebel-Zeeman topology in the total space of the fiber bundle and from it obtain a topology in the base manifold for which the continuous curves correspond to motions of charged particles in the base manifold.

### 3 Zeeman topology for Gauge Theories

Now we consider the fiber bundle structure of a Gauge Theory with a given connection as in part 2 and the Zeeman-Goebel topology for the total space of the fiber bundle. Therefore the continuous curves  $\gamma : [a, b] \rightarrow P$ , are geodesic polygonals, where geodesic means geodesic with respect to the metric given by (1).

Since we have that  $M = P/G$ , from the Zeeman-Goebel topology in  $P$  and the usual Group topology for  $G$ , we define the quotient topology in  $M$ . This is the finest topology in  $M$  that makes the projection  $\pi : P \rightarrow M$  continuous. With this topology the only continuous curves are the paths of charged particles.

For this let  $x(s)$  the motion of a particle with gauge charge  $Q$ . We take the unique geodesic lift of that curve satisfying  $\omega(\gamma'(s)) = Q$ , since it is geodesic,  $\gamma(s)$  and therefore  $x(s)$  is continuous.

If  $x(s)$  is a continuous curve in  $M$ , then for a local section  $\sigma_\alpha$ ,  $\sigma_\alpha x$  is a continuous curve  $\gamma(s)$  in the principal bundle such that  $x = \pi \circ \gamma$ , and being  $\gamma(s)$  continuous, it is a geodesic, therefore  $x = \pi \circ \gamma$  corresponds to the motion, under the given gauge field, of a particle with charge  $\omega(\gamma'(s)) = Q$ . Note that if the particle is chargeless then its path is a continuous geodesic.

If  $G = U(1)$ , the electromagnetic case, the topology we have defined works as well as Goebel topology for charged particles, being more naturally constructed and more general than this. In what follows we employ the following notation:  $Z_G(P)$  is the Zeeman - Goebel topology on  $P$  (in which continuous curves are geodesics) and  $Z_G(M)$  is the projected Zeeman - Goebel topology on  $M$  (in which the continuous curves are the paths of charged particles).

The Projected Zeeman Topology  $Z_{Pr}(M)$  is the strongest topology in which the only continuous curves are paths of charged particles

Let  $\tau^*(M)$  be a topology on  $M$  in which the only continuous curves are the paths of charged particles. We can lift this topology to a topology on  $P$  ( $\tau^*(P)$ ) as follows: Define  $\tau^*(P)$  as the weakest topology on  $P$  such that the projection  $\pi$  is continuous.

Let  $\gamma$  denote a continuous curve on  $(M, \tau^*(M))$  (and thus the path of a charged particle on  $M$ ). Using the local trivialization we can lift  $\gamma$  to a continuous path  $\gamma'$  on  $P$ . We note that  $\gamma'$  is in fact continuous in  $\tau^*(P)$  since this topology is weaker than the product topology on  $U \times G$ . Now, by the projection Theorem  $\gamma'$  is a geodesic in  $P$  and it follows that geodesics on  $P$  are continuous curves (with topology  $\tau^*(P)$ ). On the other hand, if  $\gamma'$  is a continuous curve on  $(P, \tau^*(P))$  then it projects to a continuous curve on  $(M, \tau^*(M))$ . Thus, by the projection theorem,  $\gamma'$  is a geodesic on  $P$ .

It follows that  $\tau^*(P) < Z_G(P)$  since the Zeeman - Goebel topology is the strongest topology in which the only continuous curves are geodesics.

This implies that  $\tau^*(M) < Z_{Pr}(M)$  since  $\pi$  is an open map and  $Z_{Pr}(M)$  is the strongest topology in which  $\pi$  is continuous.

We conclude then that  $Z_G(M) = Z_{Pr}(M)$ .

## 4 Conclusion

We have presented a topology for which the continuous curves are polygonals of motions of charged particles under gauge fields. If we define the physically possible movements of any massive particle as «any continuous and differentiable curve» we obtain «the possible motions in gauge fields» axiomatically from the topological structure of the space-time. This result points in the direction of a completely geometric theory of particle motion. It also suggest many directions to investigate, for example:

- (1) on what is the relation between the group of homeomorphisms of this topology and the group of gauge transformations,
- (2) what would be the generalization and interpretation of typical gauge theories ideas, as Berry's phases when the space-time has such topology,
- (3) if it is possible to obtain classification theorems of fiber bundles with a given structural group over a space-time with this topology, as it was done for the instantons.

We note that in the case of instantons classification a more pragmatic than physically acceptable employment of the four-dimensional sphere as a model for space-time was done... but never objected.

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